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ON THE PROBLEM OF BASIS FOR HYPERQUASIVARIETIES

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Abstract. Our aim is to present a solution of the Problem 31 of Chapter 8 of [5] by presenting a general theorem on hyper-quasi-identities considered as hyperbasis for solid quasi-varieties. In the last section we point out also a generalization of that result for the notion of M-hyper-quasi-identities.

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1. Notations

An identity is a pair of terms where the variables are bound by universal quantifiers. Let us take the following medial identity as an example

$$\forall u \forall x \forall y \forall w\, (u \cdot x) \cdot (y \cdot w) = (u \cdot y) \cdot (x \cdot w).$$

Let us look at the following hyperidentity

$$\forall f \forall u \forall x \forall y \forall w\, f(f(u,v), f(x,y)) = f(f(u,x), f(v,y)).$$

The hypervariable f is considered in a very specific way. Firstly every hypervariable is restricted to functional symbols of a given arity. Secondly, for a hypervariable only functions of a given arity have to be substituted, and these functions have to be term functions. Let us take the variety $A_{n,0}$ of abelian groups of finite exponent n. Every binary term $t \equiv t(x,y)$ can be presented by $t(x,y) = ax + by$ with $a, b \in \mathbb{N}_0$. If we substitute the binary hypervariable F in the above hyperidentity by $ax + by$, leaving its variables unchanged, we get

$$a(au + bv) + b(ax + by) = a(au + bx) + b(av + by).$$

This identity holds for every term $t(x,y) = ax + by$ for the variety $A_{n,0}$. Therefore we say that the hyperidentity holds for the variety $A_{n,0}$.

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2. Hyper-quasi-identities

In the sequel we avoid to use the definition of *hyperterm* from [10] and accept the convention that a *hyperterm* is formally the same as a *term*. We accept the notation of [2], [13], [7].

Hypersubstitutions of terms were defined in [4], [5] and [10]. Shortly speaking, they are mappings sending terms to terms by substituting variables by (the same) variables and functional symbols by terms of the same arities, i.e. $\sigma(x) = x$ for any variable x, and for given functional symbol f, assume that $\sigma(f)$ is a given term of the same arity as f, then σ acts on all terms of a given type in an inductive way: $\sigma(f(p_1, ..., p_n)) = \sigma(f)(\sigma(p_1), ..., \sigma(p_n))$.

$\Sigma(\tau)$ denotes the set of all hypersubstitutions σ of a given type τ.

We recall only our definitions of [10] of hyperidentities satisfied in an algebra of a given type and the notion of a *hypervariety*:

Definition 2.1. An algebra $\mathbf{A}$ *satisfies a hyperidentity* $h_1 = h_2$ if for every substitution of the hypervariables by terms (of the same arity) of $\mathbf{A}$ leaving the variables unchanged, the identities which arise hold in $\mathbf{A}$. In this case, we write $\mathbf{A} \models (h_1 = h_2)$. A variety V satisfies a hyperidentity $h_1 = h_2$ if every algebra in the variety does.

Definition 2.2. A class V of a algebras of a given type is called a *hypervariety* if and only if V is defined by a set of hyperidentities.

Remark 1. Some authors avoid to use our concepts since in [18] W. Taylor defined hypervarieties to be classes of varieties of different types satisfying certain sets of equations as identities (see also [15]).

The following was proved in [10]:

Theorem 2.3. *A variety V of type τ is defined by a set of hyperidentities if and only if $V = HSPD(V)$, i.e. V is a variety closed under derived algebras of type τ.*

We recall from [14] and [7]:

Definition 2.4. A *quasi-identity* e is an implication of the form:

$$(1.1)\ (t_0 = s_0) \wedge ... \wedge (t_{n-1} = s_{n-1}) \to (t_n = s_n).$$

where $t_i = s_i$ are k-ary identities of a given type, for $i = 0, ..., n$.

A *quasi-identity* above is *satisfied in an algebra* $\mathbf{A}$ of a given type if and only if the following implication is satisfied in $\mathbf{A}$: given a sequence $a_1, ..., a_k$ of elements of A. If these elements satisfy the equations $t_i(a_1, ..., a_k) = s_i(a_1, ..., a_n)$ in $\mathbf{A}$, for $i = 0, 1, ..., n-1$, then the equality $t_n(a_1, ..., a_k) = s_n(a_1, ..., a_k)$ is satisfied in $\mathbf{A}$. In that case we write:

$$\mathbf{A} \models (t_0 = s_0) \wedge ... \wedge (t_{n-1} = s_{n-1}) \rightarrow (t_n = s_n).$$

A quasi-identity e is *satisfied in a class* V of algebras of a given type, if and only if it is satisfied in all algebras $\mathbf{A}$ belonging to V.

Following A. I. Mal'cev [14] we consider classes QV of algebras A of a given type τ defined by quasi-identities and call them *quasivarieties*.

Formally (cf. [5], [10]), a *hyper-quasi-identity* e is the same as quasi-identity. Following the ideas of [16], part 5, cf. [5] and [11] (cf. [3]) we modify the definition above in the following way.

Definition 2.5. A hyper-quasi-identity e is *satisfied* (is *hyper-satisfied*, *holds*) in an algebra $\mathbf{A}$ if and only if the following implication is satisfied:
if σ is a hypersubstitution of type τ and the elements $a_1, ..., a_n \in A$ satisfy the equalities $\sigma(t_i)(a_1, ..., a_k) = \sigma(s_i)(a_1, ..., a_k)$ in $\mathbf{A}$, for $n = 0, 1, ..., n-1$, then the equality $\sigma(t_n)(a_1, ..., a_k) = \sigma(s_n)(a_1, ..., a_k)$ holds in $\mathbf{A}$.

In that case, we write $V \models^H e$.

In other words, hyper-quasi-identity is a universally closed Horn $\forall x \forall \sigma$-formulas, where x varies over all sequences of individual variables (occurring in terms of the implication) and σ varies over all hypersubstitutions of a given type. Our modification coincides with Definition 5.1.3 of [16] (cf. Definition 2.3 of [3]).

Remark 2. All hyper-quasi-identities and hyperidentities are written without quantifiers but they are considered as universally closed Horn $\forall$-formulas (cf. [14]). A syntactic side of the notions described here is considered in a forthcoming paper, together with a suitable generalization of the notion of hyperquasivariety to M-hyperquasivariety.

Let V be a class of algebras of type τ. Derived algebras were defined in [2]. Derived algebras of a given type τ were defined in [16], [10].

Definition 2.6. Let $\mathbf{A} = (A, F)$ be an algebra in V and σ a hypersubstitution in $\Sigma(\tau)$. Then the algebra $\mathbf{B} = (A, (F)^\sigma)$ is a *derived algebra* of $\mathbf{A}$, with the same universe A and the set $(F)^\sigma$ of all derived operations of F. We denote then $\mathbf{B}$ as $\mathbf{A}^\sigma$.

3. Problem Formulation

In the Example on p. 155 of [5] the authors proved that the quasi-identities of the given basis of [5] are hyper-quasi-identities. In the Problem 31 of Chapter 8 they posed the question:

(3.1) *Prove that it is sufficient to prove that the quasi-identities of the given basis of Example on p. 155 of* [5] *are hyper-quasi-identities for checking that every quasi-identity is a hyperquasi-identity.*

We shall refer to the question above as to the Problem (3.1).

In [5] *solid quasivarieties* were defined as quasivarieties closed under taking of derived algebras, i.e. we assume the following

Definition 3.1. Let QV be a quasivariety, then QV is *solid* if and only if every derived algebra $\mathbf{A}^{\sigma}$ belongs to QV, for every algebra $\mathbf{A}$ in QV and σ in $\Sigma(\tau)$.

We write then, that

$$QV = D(QV)$$

In [11] *solid quasivarieties* in the above sense were called *hyperquasivarieties.*

3.1. **Basis and hyperbasis.** Let Σ be a set of quasi-identities of type τ.

Definition 3.2. A quasivariety QV of all algebras of type τ satisfying all quasi-identities of Σ is called a *quasivariety defined by the basis* Σ.

Definition 3.3. A quasivariety QV of all algebras of type τ satisfying all quasi-identities of Σ as hyperquasi-identities is called a *quasivariety defined by the hyperbasis* Σ.

Recall Example of [5], p. 155:

Example. Consider the quasivariety QV of type $\tau = (2)$ defined by the following identities:

(S1) $x(yz) = (xy)z$, (identities are regarded as quasi-identities),

(S2) $xx = x$,

(S3) $(xy)(uv) = (xu)(yv)$,

(S4) $(xy = yx) \to (x = y)$.

The example above shows, that the given basis (S1)-(S4) is also a hyperbasis of QV.

The authors of [5] proved that the identities (S1), (S2), (S3) are satisfied as hyperidentities and thus as hyperquasi-identities in QV. Moreover (S4) is satisfied as a hyperquasi-identity. Finally they concluded that the quasivariety QV is solid.

Let us note, that in the proof in [5] on p. 155 of the fact above, the authors did not use their own definition on p. 155 of a *solid qusivariety.* Therefore we present here an explicit proof of the positive solution of the Problem 3.1 to make the situation more clear for the reader. Namely, our main theorem states that every quasivariety defined by a hyperbasis is solid and vice versa.

Remark 3. The positive answer of the Problem 3.1 can also be concluded via Theorem 2.6 of [3], even for the case of M-hyperquasi-equational theories. For varieties the solution can be also proved via Theorems 13.3 - 13.5 of [9] or Theorem 14.34 of [6]. In a more general setting, the solution is also implicitly contained in [6], Theorem 13.1.6 and Theorem 13.1.5 as a part of the theory of conjugate pairs of additive closure operators (cf. Lemma 4.9.2 and Theorem 4.9.3 of [5], p. 156). However, in the paper we do not use this theory, neither the authors of [3] and [6] wrote an explicit proof of the problem.

In addition, our Propositions 5.2 and 5.4 show that the Problem 3.1 considered here is mainly connected with relations between rules of inferences and not directly with the quoted above results of [3] and [5], [6].

4. Problem Solution

We present here a short proof of of the required statement by proving the following:

Main Theorem 4.1. *Let QV be a quasivariety defined by a set Σ of quasi-identities. These quasi-identities are satisfied in QV as hyper-quasi-identities if and only if QV is solid (is a hyperquasivariety). Moreover, each quasi-identity satisfied in a solid quasivariety QV is satisfied in QV as a hyper-quasi-identity and vice versa.*

Proof. Let be given a quasivariety QV with a basis Σ of quasi-identities. Assume that all quasi-identities of Σ are satisfied in QV as hyper-quasi-identities (i.e. Σ is a hyperbasis of QV). Let $\mathbf{A}$ be an algebra in QV and a derived algebra $\mathbf{B} = A^{\sigma}$. Then for any quasi-identity e of Σ, this quasi-identity e is satisfied in $\mathbf{B}$, for the following reason: each term t of type τ is realized in $\mathbf{B}$ as $\sigma(t)$, therefore for a given sequence $a_1, ..., a_k$ of elements of A, if these elements satisfy the equations $t_i(a_1, ..., a_k) = s_i(a_1, ..., a_k)$ for $i = 1, ..., n-1$, in $\mathbf{B}$ (i.e. equations $\sigma(t_i)(a_1, ..., a_k) = \sigma(s_i)(a_1, ..., a_k)$ for $i = 1, ..., n-1$ in A) then the equality $t_n(a_1, ..., a_k) = s_n(a_1, ..., a_k)$ is satisfied in $\mathbf{B}$ (i.e. the equality $\sigma(t_n)(a_1, ..., a_k) = \sigma(s_n)(a_1, ..., a_k)$ is satisfied in $\mathbf{A}$) from the assumption that Σ is a hyperbasis of QV. Therefore $\mathbf{B}$ is in QV and QV is solid.

Let QV be a solid quasivariety defined by a basis Σ. Then any quasi-identity e of Σ is satisfied in QV as a hyper-quasi-identity (i.e. Σ is a hyperbasis of QV). To show this take a σ of $\Sigma(\tau)$ and any algebra $\mathbf{A}$ of QV. Assume that for a given sequence $a_1, ..., a_k$ of elements of A, these elements satisfy the equations $\sigma(t_i)(a_1, ..., a_k) = \sigma(s_i)(a_1, ..., a_k)$ for $i = 1, ..., n-1$, then the equality $\sigma(t_n)(a_1, ..., a_k) = \sigma(s_n)(a_1, ..., a_k)$ is satisfied in the derived algebra $\mathbf{B} = \mathbf{A}^{\sigma}$, by similar arguments as above, as $\mathbf{B}$ is in QV by the assumption that QV is solid and e is satisfied in QV as an element of Σ.

Therefore we have proved that a quasivariety QV defined by a basis Σ is solid if and only if Σ is its hyperbasis. Let us note, that every quasivariety QV is defined by a basis Σ consisting of all quasi-identities satisfied in QV. This observation proves the last statement of our main theorem. □

5. Varieties

As a specific case of *quasivarieties* one may consider *varieties* of algebras and conclude a similar theorem for *basis* and *hyperbasis of identities* (cf. [10]):

Proposition 5.1. *Let V be a variety defined by a set Σ of identities. These identities are satisfied in V as hyper-identities if and only if V is solid. Moreover, each identity satisfied in a solid variety V is satisfied in V as a hyper-identity, i.e. V is a hypervariety and vice versa. Therefore the both notion coincide.*

Proof. The proof follows immediately from the Birkhoff's type theorem proved in [10]: a variety V is solid if and only if

$$V = HSPD(V).$$

□

Let us denote by E, the closure operator defined by the classical rules (1)-(5) of inferences for identities (cf. [1], [13]) and E^H denotes the closure operator defined by the rules (1) - (5) and so called *hypersubstitution rule* (6) of [10], p. 308. In [8] we proved the following Lemma 2, p. 121 (cf. Remark 1.2 of [10] p. 308):

Proposition 5.2. *If Σ is a set of identities of type τ, closed under hypersubstitution rule (6), then $E(\Sigma) = E^H(\Sigma)$.*

Let us note that the Proposition 5.2 can be slightly generalized by considering the rule $(6)_M$, i.e. the M-hypersubstitution rule of [9], p. 90, instead of the rule (6) (i.e. by considering only hypersubstitutions from a given monoid M instead all hypersubstitutions of a given type). Let E^H_M denotes the closure operator defined by the inference rules (1) - (5) of G. Birkhoff and the rule $(6)_M$. First recall from [9], p. 90:

Definition 5.3. Let be given a monoid M of hypersubstitions of type τ. The *M-hypersubstition rule* is defined by the following:

$$(6)_M \text{ from } p = q \text{ conclude } \sigma(p) = \sigma(q), \text{ for every } \sigma \in M.$$

We get:

Proposition 5.4. *If Σ is a set of identities of type τ, closed under M-hypersubstitution rule, $(6)_M$, then $E(\Sigma) = E^H_M(\Sigma)$.*

Proof. A proof is similar as those of [8], p. 121-122. The only difference is that considered hypersubstituions σ may vary only within members of M. □

Recall from [5] that hyperidentities (especially in the theory of Boolean algebras) has natural interpretations in the theory of *swiching circuts*. Therefore our observation may have some applications in algebraic computation.

6. Conclusion

By other words, we observed that solid quasivarieties are always defined by a hyperbasis. Moreover, the set of all quasi-identities satisfied in a quasivariety QV is satisfied in QV as hyperquasi-dentities if and only if QV is solid.

A suitable generalization of our observations made for the set $\Sigma(\tau)$ of all hypersubstitutions can be extended to any subset of $\Sigma(\tau)$, closed under superposition. This generalization gives rise to so called *M-hypersubstitutions* of a given type. We consider the set $\Sigma(\tau)$ of all hypersubstitutions of a given type with the superposition operation $\circ$ i.e. with the composition of hypersubstitutions) and consider the structure $(\Sigma(\tau), \circ)$ as a *monoid*. We call it the *monoid of all hypersubstitutions of a given type τ.*

Let M be a subset of $\Sigma(\tau)$ closed under superposition $\circ$, i.e. a submonoid $(M, \circ)$ of the monoid $(\Sigma(\tau), \circ)$.

Definition 6.1. A (hyper)quasi-identity e is satisfied as an *M-hyper-quasi-identity* (*is M-hypersatisfied*) in an algebra $\mathbf{A}$ if and only if the following implication is satisfied:

if σ is a hypersubstitution of type τ from the set M and the elements $a_1, ..., a_k$ of A satisfy the equalities: $\sigma(t_i)(a_1, ..., a_k) = \sigma(s_i)(a_1, ..., a_k)$ in A for $i = 1, ..., n-1$, then the equality $\sigma(t_n)(a_1, ..., a_k) = \sigma(s_n)(a_1, ..., a_k)$ holds in A.

In that case we write:

$$\mathbf{A} \models^H_M (t_0 = s_0) \wedge ... \wedge (t_{n-1} = s_{n-1}) \rightarrow (t_n = s_n).$$

Definition 6.2. Let QV be a *a quasivariety*, then QV is *M-solid* if and only if every M-derived algebra $\mathbf{A}^\sigma$ belongs to QV, for every algebra $\mathbf{A}$ in QV and σ in M.

We write then, that

$$QV = D_M(QV)$$

In [12] M-solid quasivarieties were called M-hyperquasivarieties.

We finalize our considerations by formulating a generalization of our main theorem for the case of *M-hypersubstitutions* without giving details:

Main Theorem 6.3. *Let QV be a quasivariety defined by a set Σ of quasi-identities. These quasi-identities are satisfied in QV as an M-hyper-quasi-identities if and only if QV is M-solid (is an M-hyperquasivariety). Moreover, each quasi-identity satisfied in an M-solid quasivariety QV is satisfied in QV as an M-hyper-quasi-identity and vice versa.*

Proof. A proof is similar as those of the Main Theorem 4.1. The only restriction is that hypersubstitions σ and derived algebras $\mathbf{B} = \mathbf{A}^\sigma$ are considered for all σ from a given monoid M. □

Proposition 6.4. *Let V be a variety defined by a set Σ of identities. These identities are satisfied in V as M-hyper-identities if and only if V is M-solid. Moreover, each identity satisfied in an M-solid variety V is satisfied in V as an M-hyper-identity, i.e. V is an M-hypervariety and vice versa.*

Proof. The proof follows immediately from the Birkhoff's type theorem proved in [9]: a variety V is M-solid if and only if

$$V = HSPD_M(V).$$

□

Remark 4. Let us note that in case M is a trivial (i.e. 1-element) monoid of hypersubstitutions of a given type τ, then the satisfaction $\models^H_M$ gives rise to the satisfaction $\models$ and the operator D_M to the identity operator.

In case $M = \Sigma(\tau)$ we get the notion of $\models^H$ considered in [11].

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